

# Semiclassical analysis for a Schrödinger operator with a $U(2)$ artificial gauge: the periodic case

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## Abstract

We consider a Schrödinger operator with a Hermitian  $2 \times 2$  matrix-valued potential which is lattice periodic and can be diagonalized smoothly on the whole  $R^n$ . In the case of potential taking its minimum only on the lattice, we prove that the well-known semiclassical asymptotic of first band spectrum for a scalar potential remains valid for our model.

**Keywords :** semiclassical asymptotic, spectrum, eigenvalues, Schrodinger, periodic potential, BKW method, width of the first band, magnetic field.

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## 1 Introduction

Schrödinger operators with periodic matrix-valued potentials appear in many models in physics. Such models have been used recently to describe the motion of an atom in optical fields ([Co], [Co-Da], [Da-al]), see also [Ca-Yu]. The aim of this paper is to investigate their spectral properties using semiclassical analysis. We focus on the first spectral band and assume that the potential has a non degenerate minimum. The Schrödinger operators with a non-Abelian gauge potential are Hamiltonian operators on  $L^2(\mathbb{R}^n; \mathbb{C}^m)$  of the following form :

$$H^h = h^2 \sum_{k=1}^n (D_{x_k} I - A_k)^2 + V + hQ + h^2 R = P^h(x, hD). \quad (1.1)$$

The classical symbol of  $P^h(x, hD)$ ,  $P^h(x, \xi)$ , for  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ , is given by

$$P^h(x, \xi) = \sum_{k=1}^n \{ (\xi_k I - hA_k(x))^2 + ih^2 \partial_{x_k} A_k(x) \} + V(x) + hQ(x) + h^2 R(x), \quad (1.2)$$

$I$  is the identity  $m \times m$  matrix,  $V$ ,  $Q$ ,  $R$  and the  $A_k$  are hermitian  $m \times m$  matrix with smooth coefficients and  $\Gamma$  periodic:

$$\left. \begin{aligned} A_k &= (a_{k,ij}(x))_{1 \leq i,j \leq m}, & V &= (v_{ij}(x))_{1 \leq i,j \leq m}, \\ Q &= (q_{ij}(x))_{1 \leq i,j \leq m}, & R &= (r_{ij}(x))_{1 \leq i,j \leq m}, \\ a_{k,ij}, v_{ij}, q_{ij}, r_{ij} &\in C^\infty(\mathbb{R}^n; \mathbb{C}), \\ \overline{a_{k,ji}} &= a_{k,ij}, \overline{v_{ji}} = v_{ij}, \overline{q_{ji}} = q_{ij}, \overline{r_{ji}} = r_{ij} \\ a_{k,ij}(x - \gamma) &= a_{k,ij}(x), v_{ij}(x - \gamma) = v_{ij}(x), \\ q_{ij}(x - \gamma) &= q_{ij}(x) \text{ and } r_{ij}(x - \gamma) = r_{ij}(x) \forall \gamma \in \Gamma; \end{aligned} \right\} \quad (1.3)$$

$\Gamma$  is a lattice of  $\mathbb{R}^n$ ,  $\Gamma = \left\{ \sum_{k=1}^n m_k \beta_k; m_k \in \mathbb{Z} \right\}$ ,

$\beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}^n$  form a basis,  $\det(\beta_1, \beta_2, \dots, \beta_n) \neq 0$ .

We use the notation  $D = (D_{x_1}, \dots, D_{x_n})$  where  $D_{x_k} = -i\partial_{x_k}$ ,  $k = 1 \dots n$ , so  $D^2 = -\Delta$  is the Laplacian operator on  $L^2(\mathbb{R}^n)$ .

The dual basis  $\{\beta_1^*, \dots, \beta_n^*\}$  of the reciprocal lattice  $\Gamma^*$ , is the basis of  $\mathbb{R}^n$  defined by the relations

$$\beta_j^* \cdot \beta_k = 2\pi \delta_{jk} : \quad \Gamma^* = \left\{ \sum_{k=1}^n m_k \beta_k^*; m_k \in \mathbb{Z} \right\}.$$

The fundamental cell, the Wigner-Seitz cell,

$$\mathbb{W}^n = \left\{ \sum_{k=1}^n x_k \beta_k; x_k \in \left] -\frac{1}{2}, \frac{1}{2} \right] \right\},$$

will be identified with the  $n$ -dimensional torus  $\mathbb{T}^n = \mathbb{R}^n/\Gamma$  and the dual cell, the Brillouin zone, is defined by

$$\mathbb{B}^n = \left\{ \sum_{k=1}^n \theta_k \beta_k^*; \theta_k \in ]-\frac{1}{2}, -\frac{1}{2}[ \right\}.$$

We will identify  $L^2(\mathbb{T}^n; \mathbb{C}^m)$  with  $\Gamma$  periodic functions of  $L^2_{loc}(\mathbb{R}^n; \mathbb{C}^m)$  provided with the norm of  $L^2(\mathbb{W}^n; \mathbb{C}^m)$ . In the same way the Sobolev space  $W^k(\mathbb{T}^n; \mathbb{C}^m)$ , with  $k \in \mathbb{N}$ , may be identified with  $\Gamma$  periodic functions of  $W^k_{loc}(\mathbb{R}^n; \mathbb{C}^m)$  provided with the norm of  $W^k(\mathbb{W}^n; \mathbb{C}^m)$ .

By Floquet theory, (see [Ea] or [Re-Si]), we have

$$H^h = \int_{\mathbb{B}^n}^{\oplus} H^{h,\theta} d\theta,$$

with  $H^{h,\theta}$  the partial differential operator  $P_h(x, h(D - \theta))$  on  $L^2(\mathbb{T}^n; \mathbb{C}^m)$ .

The ellipticity of  $P_h(x, h(D - \theta))$  implies that the spectrum of  $H^{h,\theta}$  is discrete

$$\text{sp}(H^{h,\theta}) = \{\lambda_j^{h,\theta}; j \in \mathbb{N}^*\}, \lambda_1^{h,\theta} \leq \lambda_2^{h,\theta} \leq \dots \leq \lambda_j^{h,\theta} \leq \lambda_{j+1}^{h,\theta} \leq \dots \quad (1.4)$$

each  $\lambda_j^{h,\theta}$  is an eigenvalue of finite multiplicity and each eigenvalue is repeated according to its multiplicity.

(When  $m = 1$  and  $V = Q = R = A_k = 0$ ,  $(\frac{1}{\sqrt{|\mathbb{T}^n|}} e^{i\omega \cdot x})_{\omega \in \Gamma^*}$  is the Hilbert basis of  $L^2(\mathbb{T}^n)$  which is composed of eigenfunctions of  $h^2(D - \theta)^2$ ).

The Floquet theory guarantees that

$$\text{sp}(H^h) = \bigcup_{\theta \in \mathbb{B}^n} \text{sp}(H^{h,\theta}) = \bigcup_{j=1}^{\infty} b_j^h, \quad (1.5)$$

where  $b_j^h$  denotes the  $j$ -th band  $b_j^h = \{\lambda_j^{h,\theta}, \theta \in \mathbb{B}^n\}$ .

In the sequel  $h_0$  will be a non negative small constant,  $h$  will be in  $]0, h_0[$ , and any non negative constant which doesn't depend on  $h$  will invariably be denoted by  $C$ .

## 2 Preliminary: the artificial gauge model

We will be interested in the model of artificial gauge considered in [Co], [Co-Da] and [Da-al]

$$\left. \begin{aligned} m = 2, \quad V = vI + W, \quad A_k = Q = R = 0, \quad \forall k, \\ W = w \cdot \sigma, \quad \text{with } w = (w_1, w_2, w_3), \quad v \text{ and the } w_j \text{ are in } C^\infty(\mathbb{R}^n; \mathbb{R}), \end{aligned} \right\} \quad (2.1)$$

we denote  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ , where the  $\sigma_j$  are the Pauli matrices.

Let us remark that

$$V = vI + W, \quad W = w.\sigma, \quad W^2 = |w|^2 I. \quad (2.2)$$

In the sequel we will assume that

$$\left. \begin{array}{l} |w| > 0 \\ v(x) - |w(x)| \text{ has a unique non degenerate minimum on } \mathbb{T}^n. \end{array} \right\} \quad (2.3)$$

Due to the invariance of the Laplacian by translation and by the action of  $\mathbb{O}(n)$ , we can assume, up to a composition by a translation of the potentials, that

$$\left. \begin{array}{l} v(\gamma) - |w(\gamma)| < v(x) - |w(x)|, \quad \forall x \in \mathbb{R}^n \setminus \Gamma \text{ and } \forall \gamma \in \Gamma, \\ v(x) - |w(x)| = E_0 + \sum_{k=1}^n \tau_k^2 x_k^2 + \mathbf{O}(|x|^3), \text{ as } |x| \rightarrow 0, \end{array} \right\} \quad (2.4)$$

( $\tau_k > 0, \forall k$ ).

There exists  $U \in \mathbb{U}(2)$ , ( a unitary  $2 \times 2$  matrix), such that

$$U^* V U = \tilde{V} = \begin{bmatrix} v - |w| & 0 \\ 0 & v + |w| \end{bmatrix}. \quad (2.5)$$

As  $|w|$  never vanishes,  $U = U(x)$  can be chosen smooth and  $\Gamma$  periodic:

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in C^\infty(\mathbb{T}^n; \mathbb{U}(2));$$

for example  $u_{11} = \frac{1}{2\sqrt{|w|(|w| - \operatorname{Re}((w_1 + iw_2)e^{-i\theta}))}}(w_3 - |w| + e^{i\theta}(w_1 - iw_2)),$

$$u_{21} = \frac{1}{2\sqrt{|w|(|w| - \operatorname{Re}(w_1 + iw_2)e^{-i\theta}))}}(w_1 + iw_2 - e^{i\theta}(w_3 + |w|)),$$

$$u_{12} = \overline{u_{21}}, \quad u_{22} = -\overline{u_{11}} \text{ and } \theta = \chi\left(\frac{w_2^2 + w_3^2}{|w|^2}\right)\frac{\pi}{2},$$

where  $\chi(t)$  is a smooth function on the real line,  $0 \leq \chi(t) \leq 1$ ,

$\chi(t) = 1$  when  $|t| \leq 1/4$  and  $\chi(t) = 0$  when  $|t| \geq 1/2$ . So

$$U = (\alpha, \beta, \rho).\sigma + i\delta\sigma_0, \quad \text{with } (\alpha, \beta, \rho, \delta) \in C^\infty(\mathbb{T}^n; \mathbb{S}^3); \quad (2.6)$$

$\sigma_0$  is the  $2 \times 2$  identity matrix and  $\mathbb{S}^3$  is the unit sphere of  $\mathbb{R}^4$ .

When  $w_1 + iw_2 \neq 0$  or when  $w_3 < 0$ , one can choose  $U$  such that  $\delta = 0$  by taking  $(\alpha, \beta, \rho) = \frac{1}{\sqrt{2|w|}}(-\frac{w_1}{\sqrt{|w| - w_3}}, -\frac{w_2}{\sqrt{|w| - w_3}}, \sqrt{|w| - w_3})$ .

Firstly let us expand the formula of the operator

$$\tilde{H}^h = U^* H^h U = h^2 D^2 I + U^* V U - 2ih^2 \sum_{k=1}^n [(U^* \partial_{x_k} U) D_{x_k} - h^2 U^* \partial_{x_k}^2 U]$$

which can be rewritten as

$$\tilde{H}^h = U^* H^h U = h^2 \sum_{k=1}^n (D_{x_k} I - A_k)^2 + \tilde{V} + h^2 R, \quad (2.7)$$

where  $A_k = iU^* \partial_{x_k} U$  :

$$A_k = [(\partial_{x_k} \alpha, \partial_{x_k} \beta, \partial_{x_k} \rho) \wedge (\alpha, \beta, \rho) + (\delta \partial_{x_k} \alpha - \alpha \partial_{x_k} \delta, \delta \partial_{x_k} \beta - \beta \partial_{x_k} \delta, \delta \partial_{x_k} \rho - \rho \partial_{x_k} \delta)] \cdot \sigma, \quad (2.8)$$

and

$$R = \sum_{k=1}^n \{ (U^* \partial_{x_k} U)^2 + (\partial_{x_k} U^*) \cdot (\partial_{x_k} U) \}. \quad (2.9)$$

So we can assume that  $H^h$  is of the form (1.1) with  $m = 2$ ,  $Q = 0$ ,  $A_k$  and  $R$  given by (2.8) and (2.9), with  $U$  defined by (2.6), and  $V = \tilde{V}$  a diagonal matrix given by (2.5).

**Theorem 2.1** *Under the above assumptions, the first bands  $b_j^h$ ,  $j = 1, 2, \dots$ , of  $H^h$  are concentrated around the value  $h\mu_j + E_0$   $j = 1, 2, \dots$ , in the sense that, there exist  $N_0 > 1$  and  $h_0 > 0$  such that*

$$\text{distance}(h\mu_j + E_0, b_j^h) \leq Ch^2, \quad \forall j < N_0 \text{ and } \forall h, 0 < h < h_0,$$

where  $\mu_j = \sum_{k=1}^n (2j_k + 1)\tau_k$ ,  $j_k \in \mathbb{N}$ , the  $(\mu_\ell)_{\ell \in \mathbb{N}^*}$  is the increasing sequence of the eigenvalues of the harmonic oscillator  $-\Delta + \sum_{k=1}^n \tau_k^2 x_k^2$ .

### 3 Proof of Theorem 2.1

*Proof.* According to the above discussion, we can assume that

$$H^h = P^h(x, hD), \quad \text{with } P^h(x, hD) = \begin{pmatrix} P_{11}^h(x, hD) & P_{12}^h(x, hD) \\ P_{21}^h(x, hD) & P_{22}^h(x, hD) \end{pmatrix}, \quad (3.1)$$

with

$$\left. \begin{aligned} P_{11}^h(x, hD) &= h^2(D - a_{\cdot,11}(x))^2 + v(x) - |w(x)| + h^2 r_{11}(x) \\ P_{22}^h(x, hD) &= h^2(D + a_{\cdot,11}(x))^2 + v(x) + |w(x)| + h^2 r_{22}(x) \\ P_{12}^h(x, hD) &= -h^2 a_{\cdot,12}(x) \cdot (D + a_{\cdot,11}(x)) - h^2 a_{\cdot,12}(x) \cdot (D - a_{\cdot,11}(x)) \\ &\quad + ih^2 \text{div}(a_{\cdot,12}(x)) + h^2 r_{12}(x) \\ P_{21}^h(x, hD) &= -h^2 a_{\cdot,21}(x) \cdot (D - a_{\cdot,11}(x)) - h^2 a_{\cdot,21}(x) \cdot (D + a_{\cdot,11}(x)) \\ &\quad + ih^2 \text{div}(a_{\cdot,21}(x)) + h^2 r_{21}(x) \end{aligned} \right\} \quad (3.2)$$

( $D = (D_{x_1}, D_{x_2}, \dots, D_{x_n})$  and  $a_{\cdot, ij}(x) = (a_{1, ij}(x), a_{2, ij}(x), \dots, a_{n, ij}(x))$ .)  
(We used that  $a_{\cdot, 22} = -a_{\cdot, 11}$ ).

Let us denote by  $H_{11}^{h, \theta}$  and  $H_{22}^{h, \theta}$  the operators associated with  $P_{11}^h(x, h(D - \theta))$  and  $P_{22}^h(x, h(D - \theta))$  on  $L^2(\mathbb{T}^n; \mathbb{C})$ .  
Then, if  $c_0 = \min |w(x)|$  and  $c_1 = \max \|R(x)\|$ ,

$$\text{sp}(H_{11}^{h, \theta}) \subset [E_0 - h^2 c_1, +\infty[ \quad \text{and} \quad \text{sp}(H_{22}^{h, \theta}) \subset [E_0 - h^2 c_1 + 2c_0, +\infty[.$$

To prove the theorem it is then enough to prove the proposition below.

**Proposition 3.1** *Let us consider a constant  $c$ ,  $0 < c < c_0$ . Then there exists  $C_0 > 0$  such that, for any  $E^h \in ]-\infty, E_0 + 2c[$ , we have*

$$\left. \begin{aligned} E^h \in \text{sp}(H_{11}^{h, \theta}) &\Rightarrow \text{distance}(E^h, \text{sp}(H_{11}^{h, \theta})) \leq C_0 h^2 \\ E^h \in \text{sp}(H_{22}^{h, \theta}) &\Rightarrow \text{distance}(E^h, \text{sp}(H_{22}^{h, \theta})) \leq C_0 h^2 \end{aligned} \right\} \quad (3.3)$$

*Proof.* For such  $E^h$ ,  $(H_{22}^{h, \theta} - E^h)^{-1}$  exists and, thanks to semiclassical pseudodifferential calculus of [Ro] (see also [Di-Sj]), for  $h_0 > 0$  small, if  $0 < h < h_0$  then  $\|(H_{22}^{h, \theta} - E^h)^{-1}\|_{L^2(\mathbb{T}^n)} + \|h(D - \theta)(H_{22}^{h, \theta} - E^h)^{-1}\|_{L^2(\mathbb{T}^n)} + \|(H_{22}^{h, \theta} - E^h)^{-1}h(D - \theta)\|_{L^2(\mathbb{T}^n)} + \|h(D - \theta)(H_{22}^{h, \theta} - E^h)^{-1}h(D - \theta)\|_{L^2(\mathbb{T}^n)} \leq C$ , and then

$$\|P_{12}^h(x, h(D - \theta))(H_{22}^{h, \theta} - E^h)^{-1}P_{21}^h(x, h(D - \theta))\|_{L^2(\mathbb{T}^n)} \leq h^2 C.$$

So if  $E^h \in \text{sp}(H_{11}^{h, \theta})$ , then  $u^h = (u_1^h, u_2^h) \neq (0, 0)$  is an eigenfunction of  $H^{h, \theta}$  associated with  $E^h$  iff

$$\left. \begin{aligned} H_{11}^{h, \theta} u_1^h + P_{12}^h(x, h(D - \theta))u_2^h &= E^h u_1^h \\ u_2^h &= -(H_{22}^{h, \theta} - E^h I)^{-1}P_{21}^h(x, h(D - \theta))u_1^h \end{aligned} \right\} \quad (3.4)$$

In fact  $E^h \in ]-\infty, E_0 + c[$  will be an eigenvalue of  $H^{h, \theta}$  iff there exists  $u_1^h$  in the Sobolev space  $W^2(\mathbb{T}^n; \mathbb{C})$ ,  $\|u_1^h\|_{L^2(\mathbb{T}^n)} \neq 0$ , such that

$$H_{11}^{h, \theta} u_1^h - P_{12}^h(x, h(D - \theta))(H_{22}^{h, \theta} - E^h I)^{-1}P_{21}^h(x, h(D - \theta))u_1^h = E^h u_1^h,$$

then we get the first part of Proposition 3.1.

If  $E^h$  is an eigenvalue of  $H_{11}^{h, \theta}$  satisfying the assumption of Proposition 3.1, and  $u_1^h$  an associated eigenfunction, then with  $u^h = (u_1^h, -(H_{22}^{h, \theta} - E^h)^{-1}P_{21}^h(x, h(D - \theta))u_1^h)$ , one has

$$\begin{aligned} &\|(H^{h, \theta} - E^h I)u^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \\ &= \|P_{12}^h(x, h(D - \theta))(H_{22}^{h, \theta} - E^h)^{-1}P_{21}^h(x, h(D - \theta))u_1^h\|_{L^2(\mathbb{T}^n; \mathbb{C})} \\ &\leq h^2 C \|u^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)}, \end{aligned}$$

we get the second part of Proposition 3.1.

Theorem 2.1 follows from Proposition 3.1 and [Si-1], [Si-2], [He-Sj-1] and [He-Sj-2] results, (see also [He]), which guarantee that the sequence of eigenvalues of  $H_{11}^{h, \theta}$ ,  $(\lambda_j(H_{11}^{h, \theta}))_{j \in \mathbb{N}^*}$  satisfies  $\forall N_0 > 1, \exists h_0 > 0, C_0 > 0$  s.t.  $\forall h, 0 < h < h_0$  and  $\forall j \leq N_0, |\lambda_j(H_{11}^{h, \theta}) - (h\mu_j^h + E_0)| \leq C_0 h^2 \square$

## 4 Asymptotic of the first band

For any real Lipschitz  $\Gamma$  periodic function  $\phi$ , and for any  $u \in W^2(\mathbb{T}^n; \mathbb{C}^2)$ , we have the identity

$$\left. \begin{aligned} & \operatorname{Re} \left( \langle P^h(x, h(D - \theta))u \mid e^{2\phi/h}u \rangle_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \right) = \\ & \sum_{k=1}^n h^2 \| ((D_{x_k} - \theta_k)I - A_k) e^{\phi/h}u \|_{L^2(\mathbb{T}^n; \mathbb{C}^2)}^2 \\ & + \langle (\tilde{V} - |\nabla \phi|^2 I + h^2 R)u \mid e^{2\phi/h}u \rangle_{L^2(\mathbb{T}^n; \mathbb{C}^2)} . \end{aligned} \right\} \quad (4.1)$$

This identity enables us to apply the method used in [He-Sj-1], (see also [He] and [Ou]). We define the Agmon [Ag] distance on  $\mathbb{R}^n$

$$d(y, x) = \inf_{\gamma} \int_0^1 \sqrt{v(\gamma(t)) - |w(\gamma(t))| - E_0} |\dot{\gamma}(t)| dt, \quad (4.2)$$

the inf is taken among paths such that  $\gamma(0) = y$  and  $\gamma(1) = x$ .

For common properties of the Agmon distance, one can see for example [Hi-Si].

We will use that, for any fixed  $y \in \mathbb{R}^n$ , the function  $d(y, x)$  is a Lipschitz function on  $\mathbb{R}^n$  and  $|\nabla_x d(y, x)|^2 \leq v(x) - |w(x)| - E_0$  almost everywhere on  $\mathbb{R}^n$ .

Using that the zeros of  $v(x) - w(x) - E_0$  are the elements of  $\Gamma$  and are non degenerate, we get that the real function  $d_0(x) = d(0, x)$  satisfies, (see [He-Sj-1]),  $|\nabla d_0(x)|^2 = v(x) - |w(x)| - E_0$  in a neighbourhood of 0.

We summarize the properties of the Agmon distance we will need:

$$\left. \begin{aligned} i) & \quad \exists R_0 > 0 \text{ s.t. } d_0(x) \in C^\infty(B_0(R_0)) \\ ii) & \quad |\nabla d_0(x)|^2 = v(x) - |w(x)| - E_0, \quad \forall x \in B_0(R_0) \\ iii) & \quad |\nabla d_0(x)|^2 \leq v(x) - |w(x)| - E_0 \\ iv) & \quad |\nabla d_\Gamma(x)|^2 \leq v(x) - |w(x)| - E_0 \end{aligned} \right\} \quad (4.3)$$

where  $d_0(x) = d(0, x)$ ,  $B_0(r) = \{x \in \mathbb{R}^n; d_0(x) < r\}$

and  $d_\Gamma(x) = d(\Gamma, x) = \min_{\omega \in \Gamma} d(\omega, x)$ .

The least Agmon distance in  $\Gamma$  is

$$S_0 = \inf_{1 \leq k \leq n} d_0(\beta_k) = \inf_{\rho \neq \omega, (\omega, \rho) \in \Gamma^2} d(\omega, \rho). \quad (4.4)$$

The Agmon distance on  $\mathbb{T}^n$ ,  $d^{\mathbb{T}^n}(\cdot, \cdot)$ , is defined by its  $\Gamma$ -periodic extension on  $(\mathbb{R}^n)^2$

$$d^{\mathbb{T}^n}(y, x) = \min_{\omega \in \Gamma} d(y, x + \omega).$$

Then

$$\frac{S_0}{2} = \sup_r \{r > 0 \text{ s.t. } \{x \in \mathbb{T}^n; d^{\mathbb{T}^n}(x_0, x) < r\} \text{ is simply connected}\}, \quad (4.5)$$

where  $x_0$  is the single point in  $\mathbb{T}^n$  such that  $v(x_0) - |w(x_0)| = E_0$ . The  $\Gamma$ -periodic function on  $\mathbb{R}^n$ ,  $d_\Gamma(x)$  is the one corresponding to the extension of  $d^{\mathbb{T}^n}(x_0, x)$ .

If  $\lambda^{h,\theta}$  is an eigenvalue of  $H^{h,\theta}$  and if  $u^{h,\theta}$  is an associated eigenfunction, then by (4.1) one gets as in the scalar case considered in [He-Sj-1] and [He-Sj-2],

$$\left. \begin{aligned} & \sum_{k=1}^n h^2 \|((D_{x_k} - \theta_k)I - A_k)e^{\phi/h} u^{h,\theta}\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)}^2 \\ & + < [\tilde{V} - |\nabla\phi|^2 I + h^2 R - \lambda^{h,\theta} I]_+ u^{h,\theta} | e^{2\phi/h} u^{h,\theta} >_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \\ & = < [\tilde{V} - |\nabla\phi|^2 I + h^2 R - \lambda^{h,\theta} I]_- u^{h,\theta} | e^{2\phi/h} u^{h,\theta} >_{L^2(\mathbb{T}^n; \mathbb{C}^2)}, \end{aligned} \right\} \quad (4.6)$$

so, when  $\phi(x) = d^{\mathbb{T}^n}(x_0, x)$ , necessarily  $\lambda^{h,\theta} - E_0 + \mathbf{O}(h^2) > 0$ , and if  $h/C < \lambda^{h,\theta} - E_0 < hC$ , then  $u^{h,\theta}$  is localized in energy near  $x_0$ , for any  $\eta \in ]0, 1[$ ,  $\exists C_\eta > 0$  such that

$$\left. \begin{aligned} & \sum_{k=1}^n h^2 \|((D_{x_k} - \theta_k)I - A_k)e^{\eta d^{\mathbb{T}^n}(x_0, x)/h} u^{h,\theta}\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)}^2 \\ & + (1 - \eta^2) < (v - |w| - E_0) u^{h,\theta} | e^{2\eta d^{\mathbb{T}^n}(x_0, x)/h} u^{h,\theta} >_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \\ & \leq hC_\eta \int_{\{x \in \mathbb{T}^n; d^{\mathbb{T}^n}(x_0, x) < \sqrt{h}C\}} |u^{h,\theta}(x)|^2 dx. \end{aligned} \right\} \quad (4.7)$$

Let  $\Omega \subset \mathbb{T}^n$  an open and simply connected set with smooth boundary satisfying, for some  $\eta$ ,  $0 < \eta < S_0/2$ ,

$$\{x \in \mathbb{T}^n; d^{\mathbb{T}^n}(x_0, x) < \frac{S_0 - \eta}{2}\} \subset \Omega \subset \{x \in \mathbb{T}^n; d^{\mathbb{T}^n}(x_0, x) < S_0/2\} \quad (4.8)$$

Let  $H_\Omega^h$  be the selfadjoint operator on  $L^2(\Omega; \mathbb{C}^2)$  associated with  $P^h(x, hD)$  with Dirichlet boundary condition. We denote by  $(\lambda_j(H_\Omega^h))_{j \in \mathbb{N}^*}$  the increasing sequence of eigenvalues of  $H_\Omega^h$ . Using the method of [He-Sj-1], we get easily the following results

**Theorem 4.1** *For any  $\eta$ ,  $0 < \eta < S_0/2$ , there exist  $h_0 > 0$  and  $N_0 > 1$  such that, if  $0 < h < h_0$  and  $j \leq N_0$ , then*

$$\forall \theta \in \mathbb{B}^n, \quad 0 < \lambda_j(H_\Omega^h) - \lambda_j^{h,\theta} \leq Ce^{-(S_0 - \eta)/(2h)}; \quad (4.9)$$

so the length of the band  $b_j^h$  satisfies  $|b_j^h| \leq Ce^{-(S_0 - \eta)/(2h)}$ .

For the first band, we have the following improvement

$$|b_1^h| \leq Ce^{-(S_0 - \eta)/h}. \quad (4.10)$$

*Sketch of the proof.*

As  $\Omega$  is simply connected and the one form  $\theta dx$  is closed, there exists a smooth real function  $\psi_\theta(x)$  on  $\overline{\Omega}$  such that  $e^{-i\psi_\theta(x)} P^h(x, hD) e^{i\psi_\theta(x)} = P^h(x, h(D - \theta))$ : the Dirichlet operators on  $\Omega$ ,  $H_\Omega^h$  and  $H_\Omega^{h,\theta}$  associated to  $P^h(x, hD)$  and  $P^h(x, h(D - \theta))$  are gauge equivalent, so they have the same spectrum.



Therefore the min-max principle says that

$$0 < \lambda_j(H_\Omega^{h,\theta}) - \lambda_j^{h,\theta} = \lambda_j(H_\Omega^h) - \lambda_j^{h,\theta}.$$

But the exponential decay of the eigenfunction  $\varphi_j^{h,\theta}(x)$  associated with  $\lambda_j^{h,\theta}$ , given by (4.7) implies that

$$\|(P^h(x, h(D - \theta)) - \lambda_j^{h,\theta})\chi\varphi_j^{h,\theta}(x)\|_{L^2(\Omega; \mathbb{C}^2)} \leq Ce^{-(S_0 - \eta + \epsilon)/(2h)},$$

for some  $\epsilon > 0$ , and for a smooth cut-off function  $\chi$  supported in  $\Omega$  and  $\chi(x) = 1$  if  $d^{\mathbb{T}^n}(x_0, x) \leq (S_0 - \eta + \epsilon)/2$ .

So  $\text{distance}(\lambda_j^{h,\theta}, \text{sp}(H_\Omega^h)) \leq Ce^{-(S_0 - \eta + \epsilon)/(2h)}$ .

This achieves the proof of (4.9).

Let us denote  $E^{h,\theta}$ , (respectively  $E^{h,\Omega}$ ), the first eigenvalue  $\lambda_1^{h,\theta}$ , (respectively  $\lambda_1(H_\Omega^h)$ ), and  $\varphi^{h,\theta}(x)$ , (respectively  $\varphi^{h,\Omega}(x)$ ), the associated normalized eigenfunctions. Let  $\chi$  be a cut-off function satisfying the same properties as before. Then  $P^h(x, h(D - \theta))(e^{-i\psi_\theta(x)}\chi(x)\varphi^{h,\Omega}(x)) = \lambda_1(H_\Omega^h)e^{-i\psi_\theta(x)}\chi(x)\varphi^{h,\Omega}(x) + e^{-i\psi_\theta(x)}r_0^h(x)$  with, thanks to the same identity as (4.7) for Dirichlet problem on  $\Omega$ ,

$$\|r_0^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \leq Ce^{-\frac{S_0 - \eta}{2h}}.$$

The same argument used in [He-Sj-1], (see also [He]), gives this estimate

$$|E^{h,\theta} - E^{h,\Omega} - \langle r_0^h | \chi\varphi^{h,\Omega} \rangle_{L^2(\mathbb{T}^n; \mathbb{C}^2)}| \leq Ce^{-(S_0 - \eta)/h}.$$

As  $\tau_h = \langle r_0^h | \chi\varphi^{h,\Omega} \rangle_{L^2(\mathbb{T}^n; \mathbb{C}^2)}$  does not depend on  $\theta$ , so

$$\forall \theta \in \mathbb{B}^n, \quad |E^{h,\theta} - E^{h,\Omega} - \tau_h| \leq Ce^{-(S_0 - \eta)/h}$$

this estimate ends the proof of (4.10)  $\square$

As for the tunnel effect in [He-Sj-1] and [Si-2], we have an accuracy estimate for the first band, like the scalar case in [Si-3] and in [Ou] (see also [He]).

**Theorem 4.2** *There exists  $h_0 > 0$  such that, if  $0 < h < h_0$  then*

$$|b_1^h| \leq Ce^{-S_0/h}.$$

*Sketch of the proof.* Instead of comparing  $H^{h,\theta}$  with an operator defined in a subset of  $\mathbb{T}^n$ , we have to work on the universal cover  $\mathbb{R}^n$  of  $\mathbb{T}^n$ .

We take  $\Omega \subset \mathbb{R}^n$  an open and simply connected set with smooth boundary satisfying, for some  $\eta_0$ ,  $0 < \eta_0 < \eta_1 < S_0/2$ ,

$$B_0((S_0 + \eta_0)/2) \subset \Omega \subset B_0((S_0 + \eta_1)/2). \quad (4.11)$$

So  $\Omega$  contains the Wigner set  $\mathbb{W}^n$ , more precisely

$$\mathbb{W}^n \subset \Omega \subset 2\mathbb{W}^n \quad \text{and} \quad \Omega \cap \Gamma = \{0\}.$$

We let also denote  $H_\Omega^h$  the Dirichlet operator on  $L^2(\Omega; \mathbb{C}^2)$  associated with  $P^h(x, hD)$ , and  $E_\Omega^h$  its first eigenvalue. The associated eigenfunction is also denoted by  $\varphi^{h, \Omega}(x)$ .

In the same way as to get (4.7), we have

$$\sum_{k=1}^n h^2 \|(D_{x_k} I - A_k) e^{d_0(x)/h} \varphi^{h, \Omega}\|_{L^2(\Omega; \mathbb{C}^2)}^2 \leq hC \int_{B_0(\sqrt{h}C)} |\varphi^{h, \Omega}(x)|^2 dx, \quad (4.12)$$

then the Poincaré estimate gives

$$\int_{\Omega} e^{2d_0(x)/h} |\varphi^{h, \Omega}(x)|^2 dx \leq h^{-1} C \int_{B_0(\sqrt{h}C)} |\varphi^{h, \Omega}(x)|^2 dx. \quad (4.13)$$

Let  $\chi$  a smooth cut-off function satisfying

$$\chi(x) = 1 \text{ if } d_0(x) \leq (S_0 + \eta_0)/2 \quad \text{and} \quad \chi(x) = 0 \text{ if } x \notin \Omega.$$

Then the function

$$\varphi^{h, \theta}(x) = \sum_{\omega \in \Gamma} e^{i\theta(x-\omega)} \chi(x-\omega) \varphi^{h, \Omega}(x-\omega)$$

is  $\Gamma$ -periodic and satisfies

$$\left. \begin{aligned} (P^h(x, h(D-\theta)) - E_\Omega^h) \varphi^{h, \theta}(x) &= r^{h, \theta} \text{ and} \\ \|r^{h, \theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)} &\leq C e^{-(S_0 + \eta_0)/(2h)} \|\varphi^{h, \theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)} \end{aligned} \right\} \quad (4.14)$$

and

$$\langle r^{h, \theta} | \varphi^{h, \theta} \rangle_{L^2(\mathbb{W}^n; \mathbb{C}^2)} =$$

$$\sum_{\omega, \rho \in \Gamma_0} e^{i\theta(\rho-\omega)} \int_{\mathbb{W}^n} ([P^h(x, hD); \chi] \varphi^{h, \Omega})(x-\omega) \cdot \overline{(\chi \varphi^{h, \Omega})}(x-\rho) dx$$

with  $\Gamma_0 = \{0, \pm\beta_1, \dots, \pm\beta_n\}$  and

$$[P^h(x, hD); \chi] = -2h^2 i \sum_{k=1}^n \partial_{x_k} \chi (D_{x_k} I - A_k) - h^2 \Delta \chi I.$$

So

$$\left| \frac{1}{\|\varphi^{h, \theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2} \langle r^{h, \theta} | \varphi^{h, \theta} \rangle_{L^2(\mathbb{W}^n; \mathbb{C}^2)} \right| \leq C e^{-S_0/h}. \quad (4.15)$$

The proof comes easily from (4.12) and (4.13) as in [Ou] or in [He].

Using the same argument of [He-Sj-1] as in the proof of (4.10), we get that

$$|E^{h, \theta} - E_\Omega^h - \tau^{h, \theta}| \leq C e^{-(S_0 + \eta_0)/h}, \quad (4.16)$$

$$\text{with } \tau^{h, \theta} = \frac{1}{\|\varphi^{h, \theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2} \langle r^{h, \theta} | \varphi^{h, \theta} \rangle_{L^2(\mathbb{W}^n; \mathbb{C}^2)}.$$

Theorem 4.2 follows from (4.15) and (4.16)  $\square$

## 5 B.K.W. method for the Dirichlet ground state

Let  $\Omega$  be an open set satisfying (4.8). more precisely  $\Omega \subset \mathbb{R}^n$  an open, bounded and simply connected set with smooth boundary satisfying, for some  $\eta_1$  and  $\eta_2$ ,  $0 < \eta_1 < \eta_2 < S_0/2$ ,

$$\{x \in \mathbb{R}^n; d_0(x) < \frac{S_0 - \eta_2}{2}\} \subset \Omega \subset \{x \in \mathbb{R}^n; d_0(x) < \frac{S_0 - \eta_1}{2}\} \quad (5.1)$$

**Theorem 5.1** *The first eigenvalue  $E^{h,\Omega} = \lambda_1(H_\Omega^h)$  of the Dirichlet operator  $H_\Omega^h$  admits an asymptotic expansion of the form*

$$E^{h,\Omega} \simeq \sum_{j=0}^{\infty} h^j e_j ,$$

and if  $S_0 - \eta_1$  is small enough, the associated eigenfunction  $\varphi^{h,\Omega}$  has also an asymptotic expansion of the form

$$\varphi^{h,\Omega} = e^{-\phi/h}(f_h^+, f_h^-) , \quad f_h^\pm \simeq \sum_{j=0}^{\infty} h^j f_j^\pm , \quad (f_0^- = 0) .$$

As usual

$$e_0 = E_0 , \quad e_1 = \tau_1 , \quad e_2 = r_{11}(0) + \sum_{k=1}^n |a_{k,11}(0)|^2 , \quad (5.2)$$

and  $\phi$  is the real function satisfying the eikonal equation

$$|\nabla \phi(x)|^2 = v(x) - |w(x)| - E_0 , \quad (5.3)$$

equal to  $d(x)$  in a neighbourhood of 0.

( $r_{11}$  and the  $a_{k,11}$  are defined by (1.1) and (1.3).  $E_0$  and  $\tau_1$  are defined by (2.2) and (2.4)).

*Proof.* When the gauge potential matrix is identified with the one form

$$A dx = \sum_{k=1}^n A_k(x) dx_k ,$$

its curvature form appears to be the related magnetic field  $B = dA + A \wedge A$  :

$$B = \sum_{1 \leq j < k \leq n} (\partial_{x_j} A_k(x) - \partial_{x_k} A_j(x)) dx_j \wedge dx_k + \sum_{1 \leq j < k \leq n} (A_j(x) A_k(x) - A_k(x) A_j(x)) dx_j \wedge dx_k .$$

For our purpose, only the vector magnetic potential  $a_{.,11}$  is significant. We will work with Coulomb vector gauge  $a_{.,11}$  :

$$\operatorname{div}(a_{.,11}(x)) = \sum_{k=1}^n \partial_{x_k} a_{k,11}(x) = 0 . \quad (5.4)$$

It is feasible thanks to the existence of a smooth real and  $\Gamma$  periodic function  $\psi(x)$  such that  $\Delta\psi(x) = \text{div}(a_{.,11}(x))$ .

Let  $\mathcal{O}$  be any open set of  $\mathbb{R}^n$  (or one can take also  $\mathcal{O} = \mathbb{T}^n$ ). Conjugation of  $P^h(x, hD)$  by the unitary operator  $J_\psi$  on  $L^2(\mathcal{O}; \mathbb{C}^2)$  :

$$J_\psi = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}, \quad (5.5)$$

leads to changing  $a_{.,11}(x)$  for  $a_{.,11}(x) - \nabla\psi(x)$  and  $a_{.,21}(x)$  for  $e^{2i\psi(x)}a_{.,21}(x)$ ; the new  $a_{.,22}(x)$  is equal to minus the new  $a_{.,11}(x)$ , and the new  $a_{.,12}(x)$  remains the conjugate of the new  $a_{.,21}(x)$ . So by (2.8) we have

$$\left. \begin{aligned} a_{.,11} &= \beta\nabla\alpha - \alpha\nabla\beta + \delta\nabla\rho - \rho\nabla\delta - \nabla\psi, \\ a_{.,21} &= e^{2i\psi}[(\rho\nabla\beta - \beta\nabla\rho + \delta\nabla\alpha - \alpha\nabla\delta) + i(\alpha\nabla\rho - \rho\nabla\alpha + \delta\nabla\beta - \beta\nabla\delta)] \\ a_{.,22} &= -a_{.,11}, \quad a_{.,12} = \overline{a_{.,21}} \\ \Delta\psi &= \text{div}(\beta\nabla\alpha - \alpha\nabla\beta + \delta\nabla\rho - \rho\nabla\delta) \end{aligned} \right\} \quad (5.6)$$

Let us write

$$e^{-\phi/h}P^h(x, hD)(e^{-\phi/h}f_h) = W_0(x)f_h + hW_1(x, D)f_h + h^2W_2(x, D)f_h \quad (5.7)$$

with

$$\begin{aligned} W_0(x) &= V(x) - |\nabla\phi(x)|^2 I \\ W_1(x, D) &= \Delta\phi I + 2i \sum_{k=1}^n \partial_{x_k} \phi (D_{x_k} I - A_k) \\ W_2(x, D) &= \sum_{k=1}^n (D_{x_k} I - A_k)^2 + R(x). \end{aligned}$$

So

$$W_1(x, D) = \begin{pmatrix} \Delta\phi - 2i\nabla\phi \cdot (i\nabla + a_{.,11}) & -2i\nabla\phi \cdot \overline{a_{.,21}} \\ -2i\nabla\phi \cdot a_{.,21} & \Delta\phi - 2i\nabla\phi \cdot (i\nabla - a_{.,11}) \end{pmatrix},$$

and

$$W_2(x, D) = \begin{pmatrix} (i\nabla + a_{.,11})^2 + r_{11} & (i\nabla + a_{.,11}) \cdot \overline{a_{.,21}} + \overline{a_{.,21}} \cdot (i\nabla - a_{.,11}) + r_{12} \\ a_{.,21} \cdot (i\nabla + a_{.,11}) + (i\nabla + a_{.,11}) \cdot a_{.,21} + r_{21} & (i\nabla - a_{.,11})^2 + r_{22} \end{pmatrix}.$$

We look for an eigenvalue  $E^h \simeq \sum_{j=0}^{\infty} h^j e_j$

and an associated eigenfunction  $f_h \simeq \sum_{j=0}^{\infty} h^j f_j$ , so

$$e^{\phi/h}(P^h(x, hD) - E^h I)(e^{-\phi/h}f_h) \simeq \sum_{j=0}^{\infty} h^j \kappa_j$$

with

$$\begin{aligned}
\kappa_0(x) &= (W_0(x) - e_0 I) f_0(x) \\
\kappa_1(x) &= (W_1(x, D) - e_1 I) f_0(x) + (W_0(x) - e_0 I) f_1(x) \\
\kappa_2(x) &= (W_2(x, D) - e_2 I) f_0(x) + (W_1(x, D) - e_1 I) f_1(x) + (W_0(x) - e_0 I) f_2(x) \\
\kappa_j(x) &= -e_j f_0(x) - \sum_{\ell=1}^{j-3} e_{j-\ell} f_\ell(x) + (W_2(x, D) - e_2 I) f_{j-2}(x) \\
&\quad + (W_1(x, D) - e_1 I) f_{j-1}(x) + (W_0(x) - e_0 I) f_j(x), \quad (j > 2).
\end{aligned}$$

We recall that  $f_j = (f_j^+, f_j^-)$  and we want that  $\kappa_j(x) = 0, \forall j$ .

### 1) Term of order 0

$$\text{As } \kappa_0(x) = 0 \Leftrightarrow \begin{cases} (-|\nabla\phi(x)|^2 + v(x) - |w(x)| - e_0) f_0^+(x) = 0 \\ (-|\nabla\phi(x)|^2 + v(x) + |w(x)| - e_0) f_0^-(x) = 0, \end{cases}$$

choosing  $\phi$  satisfying (5.3), then

$e_0 = E_0$  and  $-|\nabla\phi(x)|^2 + v(x) + |w(x)| - e_0 = 2|w(x)| > 0$  implies that

$$f_0^-(x) = 0, \quad e_0 = E_0 \quad \text{and} \quad W_0(x) - e_0 I = \begin{pmatrix} 0 & 0 \\ 0 & 2|w(x)| \end{pmatrix}. \quad (5.8)$$

### 2) Term of order 1.

The components of  $\kappa_1(x) = (\kappa_1^+(x), \kappa_1^-(x))$  become

$$\begin{aligned}
\kappa_1^+(x) &= (\Delta\phi(x) - e_1 - 2ia_{.,11}(x) \cdot \nabla\phi(x)) f_0^+(x) + 2\nabla\phi(x) \cdot \nabla f_0^+(x) \\
\kappa_1^-(x) &= 2|w(x)| f_1^-(x) - 2i(a_{.,21}(x) \cdot \nabla\phi(x)) f_0^+(x),
\end{aligned}$$

As  $|\nabla\phi(x)|$  has a simple zero at  $x_0 = 0$ , the equation  $\kappa_1^+(x) = 0$  can be solved only when  $e_1 = \Delta\phi(0)$ . In this case there exists a unique function  $f_0^+(x)$  such that  $f_0^+(0) = 1$  and  $\kappa_1^+(x) = 0$ . We can conclude that the study of the term of order 1 leads to

$$\left. \begin{aligned} e_1 &= \Delta\phi(0), \\ 2\nabla\phi(x) \cdot \nabla f_0^+(x) &= [e_1 - \Delta\phi(x) + 2ia_{.,11}(x) \cdot \nabla\phi(x)] f_0^+(x), \\ f_1^-(x) &= \frac{i}{|w(x)|} (\nabla\phi(x) \cdot a_{.,21}(x)) f_0^+(x). \end{aligned} \right\} \quad (5.9)$$

### 3) Term of order 2.

The components of  $\kappa_2(x) = (\kappa_2^+(x), \kappa_2^-(x))$  become

$$\left. \begin{aligned} \kappa_2^+(x) &= ((i\nabla + a_{.,11})^2 + r_{11}(x) - e_2) f_0^+(x) \\ &\quad + (\Delta\phi(x) - e_1 - 2ia_{.,11}(x) \cdot \nabla\phi(x)) f_1^+(x) + 2\nabla\phi(x) \cdot \nabla f_1^+(x) \\ &\quad - 2i(\overline{a_{.,21}(x)} \cdot \nabla\phi(x)) f_1^-(x) \\ \kappa_2^-(x) &= 2|w(x)| f_2^-(x) - 2i(a_{.,21}(x) \cdot \nabla\phi(x)) f_1^+(x) \\ &\quad + 2\nabla\phi(x) \cdot \nabla f_1^-(x) + 2ia_{.,11}(x) \cdot \nabla\phi(x) f_1^-(x) + 2ia_{.,21}(x) \cdot \nabla f_0^+(x) \\ &\quad + i(\text{div}(a_{.,21}(x)) f_0^+(x) + r_{21}(x) f_0^+(x)), \end{aligned} \right\} \quad (5.10)$$

The unknown function  $f_1^+(x)$  must give  $\kappa_2^+(x) = 0$  in (5.10). This equation, with the initial condition  $f_1^+(0) = 0$ , can be solved only if

$$e_2 = (i\nabla + a_{.,11})^2 f_0^+(0) + r_{11}(0).$$

(We used that  $f_0^+(0) = 1$ ). So  $\kappa_2 = 0$  implies

$$\begin{aligned} e_2 &= (i\nabla + a_{.,11})^2 f_0^+(0) + r_{11}(0), \\ 2\nabla\phi(x) \cdot \nabla f_1^+(x) &= -(\Delta\phi(x) - e_1 - 2ia_{.,11}(x) \cdot \nabla\phi(x)) f_1^+(x) \\ &\quad - ((i\nabla + a_{.,11})^2 + e_2 - r_{11}(x)) f_0^+(x) + 2i(a_{.,12}(x) \cdot \nabla\phi(x)) f_1^-(x) \\ f_2^-(x) &= \frac{1}{2|w(x)|} [2i(a_{.,21}(x) \cdot \nabla\phi(x)) f_1^+(x) - 2\nabla\phi(x) \cdot \nabla f_1^-(x) - 2i(a_{.,11}(x) \cdot \nabla\phi(x)) f_1^-(x) \\ &\quad - 2ia_{.,21}(x) \cdot \nabla f_0^+(x) - i(\operatorname{div}(a_{.,21}(x))) f_0^+(x) - r_{21}(x) f_0^+(x)]. \end{aligned}$$

#### 4) Term of order $j > 2$ .

We assume that  $e_\ell$  for  $\ell = 0, 1, \dots, j-1$ , the functions  $f_\ell^\pm(x)$  for  $\ell = 0, 1, \dots, j-2$ , and the one  $f_{j-1}^-(x)$  are well-known,  $f_\ell^+(0) = 0$  when  $0 < \ell < j-1$ .

The equation  $\kappa_j^+ = 0$  becomes

$$\left. \begin{aligned} 2\nabla\phi(x) \cdot \nabla f_{j-1}^+(x) + (\Delta\phi(x) - e_1 - 2ia_{.,11}(x) \cdot \nabla\phi(x)) f_{j-1}^+(x) &= \\ 2i(\overline{a_{.,21}(x)} \cdot \nabla\phi(x)) f_{j-1}^-(x) + \sum_{\ell=0}^{j-3} e_{j-\ell} f_\ell^+(x) & \\ -((i\nabla + a_{.,11})^2 - e_2 + r_{11}(x)) f_{j-2}^+(x) & \\ -(r_{12}(x) + i\operatorname{div}(\overline{a_{21}})) f_{j-2}^-(x) - 2i\overline{a_{21}} \cdot \nabla f_{j-2}^-(x) & \end{aligned} \right\} \quad (5.11)$$

This equation has a unique solution  $f_{j-1}^+(x)$  such that  $f_{j-1}^+(0) = 0$  iff

$$\left. \begin{aligned} e_j &= ((i\nabla + a_{.,11})^2 + r_{11}(0)) f_{j-2}^+(0) + r_{12}(0) f_{j-2}^-(0) \\ -2ia_{.,21}(0) \cdot \nabla f_{j-2}^-(0) - \sum_{\ell=0}^{j-3} e_{j-\ell} f_\ell^+(0) & \end{aligned} \right\} \quad (5.12)$$

The equation  $\kappa_j^- = 0$  gives

$$f_j^-(x) = \frac{1}{2|w(x)|} \times$$

$$\begin{aligned} &[-2\nabla\phi(x) \cdot \nabla f_{j-1}^-(x) + (e_1 - 2ia_{.,11}(x) \cdot \nabla\phi(x)) f_{j-1}^-(x) + 2ia_{.,21}(x) \cdot \nabla\phi(x) f_{j-1}^+(x) \\ &+ (e_2 - (i\nabla - a_{.,11})^2) f_{j-2}^-(x) - i\operatorname{div}(a_{.,21}(x)) f_{j-2}^+(x) - 2ia_{.,21}(x) \cdot \nabla f_{j-2}^+(x) + \sum_{\ell=0}^{j-3} e_{j-\ell} f_\ell^-(x)] \end{aligned}$$

#### 5) End of the proof.

Let  $\chi(x)$  be a cut-off function equal to 1 in a neighbourhood of 0 and supported in  $\Omega$ . Then taking  $\chi(x)f_j(x)$  instead of  $f_j(x)$ , we get a function  $\varphi^{h,\Omega}$  satisfying Dirichlet boundary condition and Theorem 5.1. The self-adjointness of the related operator ensures that the computed sequence  $(e_j)$  is real  $\square$

## 6 Sharp asymptotic for the width of the first band

Returning to the proof of Theorem 4.2, we have to study carefully the  $\tau^{h,\theta}$  defined in (4.16), using the method of [He-Sj-1] performed in [He] and [Ou].

Using (4.14)–(4.16), we have

$$\tau^{h,\theta} = \sum_{\omega \in \Gamma_0^+} \left( e^{-i\theta\omega} (\rho_\omega^+ + \rho_\omega^-) + e^{i\theta\omega} \overline{(\rho_\omega^+ + \rho_\omega^-)} \right), \quad (6.1)$$

with  $\Gamma_0^+ = \{\beta_1, \dots, \beta_n\}$ ,

$$\rho_\omega^+ = \frac{1}{\|\varphi^{h,\theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2} \int_{\mathbb{W}^n} ([P^h(x, hD); \chi] \varphi^{h,\Omega})(x - \omega) \cdot \overline{(\chi \varphi^{h,\Omega})}(x) dx$$

$$\text{and } \rho_\omega^- = \frac{1}{\|\varphi^{h,\theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2} \int_{\mathbb{W}^n} ([P^h(x, hD); \chi] \varphi^{h,\Omega})(x) \cdot \overline{(\chi \varphi^{h,\Omega})}(x + \omega) dx.$$

We get from the formula of  $[P^h(x, hD); \chi]$  and from the estimate (4.13) that

$$\begin{aligned} \rho_\omega^+ = & -\frac{h^2}{\|\varphi^{h,\theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2} \int_{\mathbb{W}^n} (\nabla \chi(x - \omega) \nabla \varphi^{h,\Omega})(x - \omega) \cdot \overline{(\chi \varphi^{h,\Omega})}(x) dx + \\ & + \mathbf{O}(he^{-S_0/h}) \end{aligned} \quad (6.2)$$

and

$$\rho_\omega^- = -\frac{h^2}{\|\varphi^{h,\theta}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2} \int_{\mathbb{W}^n} (\nabla \chi(x) \nabla \varphi^{h,\Omega})(x) \cdot \overline{(\chi \varphi^{h,\Omega})}(x + \omega) dx + \mathbf{O}(he^{-S_0/h}).$$

**Theorem 6.1** *Under the assumption of Theorem 4.2, if for any  $\omega \in \{\pm\beta_1, \dots, \pm\beta_n\}$  such that the Agmon distance in  $\mathbb{R}^n$  between 0 and  $\omega$  is the least one, (i.e.  $d(0, \omega) = S_0$ ), there exists one or a finite number of minimal geodesics joining 0 and  $\omega$ , then there exists  $\eta_0 > 0$  and  $h_0 > 0$  such that*

$$b_1^h = \eta_0 h^{1/2} e^{-S_0/h} (1 + \mathbf{O}(h^{1/2})) , \quad \forall h \in ]0, h_0[.$$

*Sketch of the proof.* Following the proof of splitting in [He-Sj-1] and [He], in (6.2) we can change  $\mathbb{W}^n$  into  $\mathbb{W}^n \cap \mathcal{O}$ , where  $\mathcal{O}$  is any neighbourhood of the minimal geodesics between 0 and the  $\pm\beta_k$ , such that  $d(x) = d(0, x) \in C^\infty(\mathcal{O})$ . In this case the B.K.W. method is valid in  $\mathbb{W}^n \cap \mathcal{O}$ . If  $\varphi_{B.K.W.}^h$  is the B.K.W. approximation of  $\varphi^{h,\Omega}$  in  $\mathbb{W}^n \cap \mathcal{O}$ , then, thanks to (4.1), for any  $p_0 > 0$  there exists  $C_{p_0}$  such that

$$\begin{aligned} h \sum_{k=1}^n \|(D_{x_k} I - A_k) e^{d(x)/h} \chi_0 (\varphi^{h,\Omega} - \varphi_{B.K.W.}^h)\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2 \\ + \|e^{d(x)/h} \chi_0 (\varphi^{h,\Omega} - \varphi_{B.K.W.}^h)\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2 \leq h^{p_0} C_{p_0}, \end{aligned} \quad (6.3)$$

where  $\chi_0$  is a cut-off function supported in  $\mathbb{W}^n \cap \mathcal{O}$  and equal to 1 in a neighborhood of the minimal geodesics between 0 and the  $\pm\beta_k$ . We have assumed that  $\|\varphi^{h,\Omega}\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)} = 1$  and then  $\|\chi_0 \varphi_{B.K.W.}^h\|_{L^2(\mathbb{W}^n; \mathbb{C}^2)}^2 - 1 = \mathbf{O}(h^p)$  for any  $p > 0$ .

As (6.2) remains valid if we change  $\varphi^{h,\Omega}$  into  $\chi_0\varphi^{h,\Omega}$ , the estimate (6.3) allows also to change  $\varphi^{h,\Omega}$  into  $\chi_0\varphi_{B.K.W.}^h$ . As a consequence, Theorem 6.1 follows easily, if in  $\mathbb{W}^n \cap \mathcal{O}$ ,  $\chi(x) = \chi_1(d(x))$  for a decreasing function  $\chi_1$  on  $[0, +\infty[$  with compact support, equal to 1 in a neighborhood of 0. In this case (6.2) becomes

$$\rho_{\omega}^{+} = \frac{h^{(2-n)/2}}{(2\pi)^{1/2} \prod_{k=1}^n \tau_k^{1/2}} \times \quad (6.4)$$

$$\int_{\mathbb{W}^n \cap \mathcal{O}} \chi_1'(d(x-\omega)) \chi_1(d(x)) |\nabla d(x-\omega)|^2 f_0^{+}(x-\omega) f_0^{+}(x) e^{-(d(x-\omega)+d(x))/h} dx + \mathbf{O}(he^{-S_0/h})$$

and

$$\rho_{\omega}^{-} = \frac{h^{(2-n)/2}}{(2\pi)^{1/2} \prod_{k=1}^n \tau_k^{1/2}} \times$$

$$\int_{\mathbb{W}^n \cap \mathcal{O}} \chi_1'(d(x)) \chi_1(d(x+\omega)) |\nabla d(x)|^2 f_0^{+}(x) f_0^{+}(x+\omega) e^{-(d(x)+d(x+\omega))/h} dx + \mathbf{O}(he^{-S_0/h}).$$

We remind that for any  $y$  in a minimal geodesic joining 0 to  $\pm\beta_k$ , if  $y \neq 0$  and  $y \neq \pm\beta_k$ , then the function  $d(x) + d(x \mp \beta_k)$ , when it is restricted to any hypersurface orthogonal to the geodesic through  $y$ , has a non degenerate minimum  $S_0$  at  $y$   $\square$

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